

Exact mesoscopic correlation functions of the Richardson pairing model

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We study the static correlation functions of the Richardson pairing model (also known as the reduced or discrete-state BCS model) in the canonical ensemble. Making use of the algebraic Bethe ansatz formalism, we obtain exact expressions which are easily evaluated numerically for any value of the pairing strength up to large numbers of particles. We provide explicit results at half-filling and extensively discuss their finite-size scaling behavior.

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I. INTRODUCTION

The pairing phenomenon is ubiquitous in quantum many-body systems of sizes ranging from the very small, such as quarks and nuclei, to the very large, such as stars.^{1,2} The common feature of all these seemingly unrelated systems is the instability against the formation of Cooper pairs for an arbitrarily weak attractive force, the basis of the BCS theory of superconductivity.³ Despite the diverse nature of pairing systems, many of their fundamental properties can be understood phenomenologically from a so-called reduced BCS model,

$$H_{BCS} = \sum_{\sigma=+,-} \sum_{\alpha=1}^N \frac{\epsilon_{\alpha}}{2} c_{\alpha\sigma}^{\dagger} c_{\alpha\sigma} - g \sum_{\alpha,\beta=1}^N c_{\alpha+}^{\dagger} c_{\alpha-}^{\dagger} c_{\beta-} c_{\beta+}, \quad (1)$$

which was studied by Richardson in the early 1960's in the context of nuclear physics.⁴ The model simply describes (pseudo) spin-1/2 fermions (electrons, nucleons, etc.) in a shell of doubly degenerate single particle energy levels with energies $\epsilon_{\alpha}/2$, $\alpha=1, \dots, N$. $c_{\alpha,\sigma}$ are the annihilation operators, $\sigma=+, -$ labels the degenerate time reversed states (i.e., spin or isospin), and g denotes the pairing coupling constant. Despite its simplified character (all levels interact uniformly), this Hamiltonian captures the main essence of the problem; fixing (*ab initio* or phenomenologically) the energy levels ϵ_{α} and the coupling g allows one to obtain quantitative predictions, since the model remains solvable for an arbitrary choice of parameters.

In the thermodynamic limit and within the grand-canonical ensemble, the properties of the Richardson model are correctly described via the BCS variational ansatz.³ However, for finite numbers of particles, the situation is more complex. The actual solution then depends on the ensemble chosen, and for physically relevant systems, the grand-canonical ensemble is not always the appropriate one. For example, nuclei have a fixed number of nucleons; due to the typically large charging energy, experiments on ultrasmall metallic grains are also performed at a fixed number of electrons.⁵ In these cases, a treatment based on the canonical ensemble would be more appropriate, precluding a BCS mean-field approach. Moreover, dealing with a system in the mesoscopic regime precludes the use of quantum statistical mechanics, and one is thus forced to rely either on uncon-

trolled approximations or nonperturbative methods.

Fortunately, Richardson's Hamiltonian in Eq. (1) is one of the theories for which an exact solution can be constructed in the canonical ensemble.⁴ This solution explains several interesting features of the mesoscopic physics of superconductors, complementing previous approximate treatments (see the review in Ref. 6). In particular, it allowed to give a definitive answer to Anderson's 1959 question:⁷ What is the size limit for a metallic grain to have superconducting properties? The utility of the model is thus indisputable (see also the reviews in Refs. 8 and 9 for some non-condensed-matter applications); however, most of the attention has been concentrated on thermodynamical quantities. On the other hand, experiments typically give access to static or dynamical correlation functions, which are not easily obtained in this framework. Richardson¹⁰ himself in 1965 derived a first exact expression for static correlation functions, which unfortunately has a degree of complexity that grows factorially with system size, and was therefore not suitable for actual calculations. In a significant development, Amico and Osterloh¹¹ proposed a new method (based on a generalization of earlier work by Sklyanin¹²) to write down such correlations explicitly. The complexity of this method was still factorial and all the numerical results were therefore limited to system sizes of up to 16 particles. A disadvantage of these methods is that all the eigenstates of the Hamiltonian must be known to get the correlation functions.

A major simplification was then proposed by Zhou *et al.*¹³ (see also Ref. 14). Using the algebraic Bethe ansatz (ABA) and the Slavnov formula for scalar products of states,¹⁵ they managed to write the static correlation functions as sums over N_r^2 determinants of $N_r \times N_r$ matrices, reducing the complexity of the problem to order N_r^5 (here, N_r is the number of rapidities in the eigenstates). Furthermore, in this approach, only the knowledge of the ground-state wave function is required. Surprisingly, this approach has not been used until now to obtain quantitative numerical results for the correlation functions, with the notable exception of the calculation of ground-state entanglement properties.¹⁶

In this paper, we fill this gap. As the first step, we reanalyze the results of Refs. 13 and 14, rewriting all static correlation functions as sums over only N_r determinants (thereby reducing the complexity of the problem by a further factor of N_r). We then provide analytical formulas for the physically

relevant correlation functions and evaluate them for some model Hamiltonians. We stress that having reduced the complexity of the problem by this amount, correlation functions of systems with many more particles than before can be calculated on a simple computer. In this way, we can describe the crossover from mesoscopic to macroscopic physics, going beyond previous results limited to fewer particles.^{11,17}

The paper is organized as follows. In Sec. II, we discuss the model and its general properties. In Sec. III, we recall how to calculate static correlation functions by means of the algebraic Bethe ansatz and derive their general expressions in terms of sums of N_r determinants. This section is rather technical; thus, the reader interested in the physical results can skip directly to Sec. IV, where we discuss how to solve the Richardson equations for the ground state and derive quantities that do not require knowledge of the determinant representation. In Sec. V, all the correlation functions are explicitly calculated at half-filling. The paper is closed by Sec. VI, where we also discuss open problems for future investigation.

II. MODEL

A simple but very important property of the system is the so-called blocking effect,^{4,6} i.e., unpaired particles completely decouple from the dynamics and behave as if they were free. We will denote the total number of fermions as N_f and the total number of pairs as N_p . Due to level blocking, we will only consider $N_f = 2N_p$ paired particles in N unblocked levels. In terms of pair annihilation and creation operators,

$$b_\alpha = c_{\alpha-}c_{\alpha+}, \quad b_\alpha^\dagger = c_{\alpha+}^\dagger c_{\alpha-}^\dagger, \quad (2)$$

the Hamiltonian is

$$H = \sum_{\alpha=1}^N \epsilon_\alpha b_\alpha^\dagger b_\alpha - g \sum_{\alpha,\beta=1}^N b_\alpha^\dagger b_\beta, \quad (3)$$

and $n_\alpha = 2b_\alpha^\dagger b_\alpha$ is the number of particles in level α .

The pair creation and annihilation operators satisfy the commutation relations

$$[b_\alpha, b_\beta^\dagger] = \delta_{\alpha\beta}(1 - 2b_\alpha^\dagger b_\alpha), \quad [b_\alpha, b_\beta] = [b_\alpha^\dagger, b_\beta^\dagger] = 0. \quad (4)$$

The term $2b_\alpha^\dagger b_\alpha$ in the first commutator makes the model different from free bosons, and therefore nontrivial.

Using the pseudospin realization of electron pairs, $S_\alpha^z = b_\alpha^\dagger b_\alpha - 1/2$, $S_\alpha^- = b_\alpha$, $S_\alpha^+ = b_\alpha^\dagger$, the BCS Hamiltonian becomes (up to a constant)

$$H = \sum_{\alpha=1}^N \epsilon_\alpha S_\alpha^z - g \sum_{\alpha,\beta=1}^N S_\alpha^+ S_\beta^-. \quad (5)$$

The operators $S_\alpha^{\pm,z}$ obey a standard spin algebra and so the Hamiltonian in Eq. (5) describes a spin-1/2 magnet with long-range interaction for the XY components in a site-dependent transverse magnetic field ϵ_α . Such a magnetic Hamiltonian is known in the literature as a Gaudin magnet.¹⁸ An important relation is

$$S_\alpha^+ S_\alpha^- = S_\alpha^2 - (S_\alpha^z)^2 \pm S_\alpha^z. \quad (6)$$

Since the normalization of the pairing strength in the literature is not uniform, care must be taken when comparing the results we will obtain with other papers (e.g., our g is the half of the coupling used in Refs. 11 and 17).

A. Grand-canonical BCS wave function

In the grand-canonical (GC) ensemble the ground-state wave function is the BCS variational ansatz

$$|GS\rangle = \prod_{\alpha} (u_{\alpha} + e^{i\phi_{\alpha}} v_{\alpha} b_{\alpha}^{\dagger}) |0\rangle \quad \text{with } u_{\alpha}^2 + v_{\alpha}^2 = 1, \quad (7)$$

where the variational parameters u_{α} and v_{α} are real and ϕ_{α} is a phase which, it turns out, must be α independent. $|GS\rangle$ is not an eigenstate of the particle number operator N_f and the average condition $\langle N_f \rangle = \bar{N}_f$ determines the GC chemical potential. Likewise, the commonly used definition

$$\Delta_{GC} = 2g \sum_{\alpha} \langle b_{\alpha} \rangle = 2g \sum_{\alpha} u_{\alpha} v_{\alpha} e^{i\phi_{\alpha}} \quad (8)$$

for the superconducting gap makes sense only in a GC ensemble, since $\langle b_{\alpha} \rangle$ is zero when evaluated at a fixed particle number. The variational parameters are obtained as

$$v_{\alpha}^2 = \frac{1}{2} \left[1 - \frac{\epsilon_{\alpha} - \mu}{\sqrt{(\epsilon_{\alpha} - \mu)^2 + |\Delta_{GC}|^2}} \right], \quad (9)$$

where μ is the GC chemical potential.

It is then easy to calculate (static) correlation functions on this GS:

$$\langle b_{\alpha}^{\dagger} b_{\alpha} \rangle = v_{\alpha}^2, \quad \langle b_{\alpha} b_{\alpha}^{\dagger} \rangle = u_{\alpha}^2, \quad \langle b_{\alpha}^{\dagger} b_{\beta} \rangle = u_{\alpha} v_{\alpha} u_{\beta} v_{\beta}. \quad (10)$$

B. Canonical description and Richardson solution

The exact solution of Eq. (1) was worked out by Richardson.⁴ In the canonical ensemble, the model is integrable¹⁹ and tractable by means of algebraic methods.^{13,14,20,21} We review here only the main points of this solution.

In the ABA, eigenstates are constructed by applying raising operators on a so-called reference state (pseudovacuum). Here, we choose the pseudovacuum (in the spin representation) to be fully polarized along the \hat{z} axis,

$$S_{\alpha}^z |0\rangle = \frac{1}{2} |0\rangle \quad \forall \alpha. \quad (11)$$

In the pair representation, this state thus corresponds to having one pair in each available level. Eigenstates with N_p pairs are then characterized by $N_r = N - N_p$ spectral parameters (rapidities) w_j and take the form of Bethe wave functions

$$|\{w_j\}\rangle = \prod_{k=1}^{N_r} \mathcal{B}(w_k) |0\rangle. \quad (12)$$

The operators \mathcal{B} , together with operators $\mathcal{A}, \mathcal{C}, \mathcal{D}$ defined as

$$\mathcal{A}(w_k) = \frac{-1}{g} + \sum_{\alpha=1}^N \frac{S_{\alpha}^z}{w_k - \epsilon_{\alpha}}, \quad \mathcal{B}(w_k) = \sum_{\alpha=1}^N \frac{S_{\alpha}^-}{w_k - \epsilon_{\alpha}},$$

$$\mathcal{C}(w_k) = \sum_{\alpha=1}^N \frac{S_{\alpha}^{+}}{w_k - \epsilon_{\alpha}}, \quad \mathcal{D}(w_k) = \frac{1}{g} - \sum_{\alpha=1}^N \frac{S_{\alpha}^{-}}{w_k - \epsilon_{\alpha}}, \quad (13)$$

obey the Gaudin algebra, which is the quasiclassical limit of the quadratic Yang-Baxter algebra associated with the $gl(2)$ invariant R matrix (we refer the readers to Ref. 14 for details).

The wave functions in Eq. (12) are eigenstates of the transfer matrix, and thus of the Hamiltonian in Eq. (1), when the parameters w_j satisfy the Richardson equations

$$\frac{1}{g} = \sum_{\alpha=1}^N \frac{1}{w_j - \epsilon_{\alpha}} - \sum_{k \neq j}^{N_r} \frac{2}{w_j - w_k}, \quad j = 1, \dots, N_r. \quad (14)$$

Throughout the paper, we will denote with Latin indices the rapidities and with Greek ones the energy levels. The total energy of a Bethe state is $E = \sum_{\alpha=1}^N \frac{\epsilon_{\alpha}}{2} - \sum_j w_j + g(2N_r - N)$. For a given N and N_r , the number of solutions of the Richardson equations is $\binom{N}{N_r}$ and coincides with the dimension of the Hilbert space of N_r pair vacancies distributed into N different levels, i.e., the solutions to the Richardson equations give all the eigenstates of the model.

Note that the Richardson equations in Eq. (14) have a different sign of g compared to the ones mostly considered in the literature. This is due to the particular choice of the pseudovacuum we made in Eq. (11), whereas the most common choice is $S_{\alpha}^{-}|0\rangle = -1/2|0\rangle$. With our choice of pseudovacuum, Bethe states are built by destroying pairs, as in Eq. (12) and not by creating them. We use this somehow unusual pseudovacuum following Refs. 13 and 14 in order to use all the formulas there without any adaptation. At half-filling (that is the only case considered numerically here), the different choice of the pseudovacuum only matters as a global normalization and a different labeling of the states.

The connection between the canonical and grand-canonical ensembles was first pointed out by Richardson himself,²² who showed how in the large N_f limit one recovers the BCS gap equation as

$$N_f = \sum_{\alpha=1}^N \left(1 - \frac{\epsilon_{\alpha} - \mu}{\sqrt{(\epsilon_{\alpha} - \mu)^2 + |N_f \Delta|^2}} \right), \quad (15)$$

where μ is now fixed by the density and the equation can be solved to find Δ , which, with this normalization, is an *intensive* quantity and corresponds to Δ_{GC}/N_f . For the ground-state energy per pair, E_0 , one finds

$$N_p E_0 = \sum_{\alpha=1}^N \epsilon_{\alpha} \left(1 - \frac{\epsilon_{\alpha} - \mu}{\sqrt{(\epsilon_{\alpha} - \mu)^2 + |N_f \Delta|^2}} \right) - \frac{N_f^2 \Delta^2}{2g}. \quad (16)$$

Anderson⁷ argued that by increasing the mean energy spacing d (that is inversely proportional to the volume in a metallic grain), superconductivity should disappear when d becomes of the order of the bulk gap Δ_{GC} . Our study of correlation functions (to be presented below) clearly shows this crossover.

III. ALGEBRAIC BETHE ANSATZ AND CORRELATION FUNCTIONS

The starting point to calculate correlation functions with the ABA is having a representation for the scalar products of two generic states defined by N_r rapidities ($N - N_r$ pairs),

$$\langle \{w\} | \{v\} \rangle = \langle 0 | \prod_{b=1}^{N_r} \mathcal{C}(w_b) \prod_{a=1}^{N_r} \mathcal{B}(v_a) | 0 \rangle, \quad (17)$$

when at least one set of parameters (e.g., w_b but not v_a) is a solution to the Richardson equations. Following standard notations, \mathcal{C} is the conjugate of the operator \mathcal{B} . Such a representation exists and is known as the Slavnov formula,¹⁵ which, for the case at hand, specifically reads¹³

$$\langle \{w\} | \{v\} \rangle = \frac{\prod_{a \neq b}^{N_r} (v_b - w_a)}{\prod_{b < a} (w_b - w_a) \prod_{a < b} (v_b - v_a)} \det_{N_r} J(\{v_a\}, \{w_b\}), \quad (18)$$

where the matrix elements of J are given by

$$J_{ab} = \frac{v_b - w_b}{v_a - w_b} \left(\sum_{\alpha=1}^N \frac{1}{(v_a - \epsilon_{\alpha})(w_b - \epsilon_{\alpha})} - 2 \sum_{c \neq a}^{N_r} \frac{1}{(v_a - v_c)(w_b - v_c)} \right), \quad (19)$$

from which the norms of states simply follow from $v \rightarrow w$ as $\|\{v\}\|^2 = \det_{N_r} G$ with the Gaudin matrix

$$G_{ab} = \begin{cases} \sum_{\beta=1}^N \frac{1}{(v_a - \epsilon_{\beta})^2} - 2 \sum_{c \neq a}^{N_r} \frac{1}{(v_a - v_c)^2}, & a = b \\ \frac{2}{(v_a - v_b)^2}, & a \neq b, \end{cases} \quad (20)$$

recovering Richardson's old result.¹⁰

The key point is that any form factor of a local spin operator between two Bethe eigenstates can be represented via Eq. (13) as a scalar product with one set, e.g., $\{v\}$, not satisfying the Bethe equations, for which Slavnov's formula is applicable. This has been explicitly worked out in Ref. 13. For $\{w\}$, $\{v\}$ containing, respectively, $N_r + 1$ and N_r elements, the nonzero form factors are

$$\begin{aligned} \langle \{w\} | S_{\alpha}^{-} | \{v\} \rangle &= \langle \{v\} | S_{\alpha}^{+} | \{w\} \rangle \\ &= \frac{\prod_{b=1}^{N_r+1} (w_b - \epsilon_{\alpha})}{\prod_{a=1}^{N_r} (v_a - \epsilon_{\alpha})} \frac{\det_{N_r+1} T(\alpha, \{w\}, \{v\})}{\prod_{b > a} (w_b - w_a) \prod_{b < a} (v_b - v_a)}, \end{aligned} \quad (21)$$

and, for both $\{w\}$ and $\{v\}$ containing N_r rapidities,

$$\begin{aligned} \langle \{w\} | S_{\alpha}^z | \{v\} \rangle &= \prod_{a=1}^{N_r} \frac{(w_a - \epsilon_{\alpha})}{(v_a - \epsilon_{\alpha})} \frac{\det_{N_r} \left[\frac{1}{2} T_z(\{w\}, \{v\}) - Q(\alpha, \{w\}, \{v\}) \right]}{\prod_{b>a} (w_b - w_a) \prod_{b<a} (v_b - v_a)}, \end{aligned} \quad (22)$$

with the matrix elements of T given by

$$\begin{aligned} T_{ab}(\alpha) &= \prod_{c \neq a}^{N_r+1} (w_c - v_b) \left(\sum_{\alpha=1}^N \frac{1}{(v_b - \epsilon_{\alpha})(w_a - \epsilon_{\alpha})} \right. \\ &\quad \left. - 2 \sum_{c \neq a} \frac{1}{(v_b - w_c)(w_a - w_c)} \right), \quad b < N_r + 1, \\ T_{aN_r+1}(\alpha) &= \frac{1}{(w_a - \epsilon_{\alpha})^2}, \quad Q_{ab}(\alpha) = \frac{\prod_{c \neq b} (v_c - v_b)}{(w_a - \epsilon_{\alpha})^2}. \end{aligned} \quad (23)$$

Above, T_z is the $N_r \times N_r$ matrix obtained from T by deleting the last row and column and replacing N_r+1 by N_r in the matrix elements. Here, it is assumed that both $\{v_a\}$ and $\{w_b\}$ are solutions to Richardson's Bethe equations. However, the results are still valid for S_{α}^{\pm} if only $\{w_b\}$ satisfy the Bethe equations.

A. Determinant representation of the correlation functions

In Ref. 13, it has been pointed out that due to the simplicity of the solution of the ABA, not only the form factors but any static correlation function can be written in a determinant representation. This simplicity puts the BCS model in an extremely privileged position for a detailed study of the static correlation functions.

The result for $\langle \{w\} | S_{\alpha}^+ S_{\beta}^+ | \{v\} \rangle$ has been explicitly worked out,¹³

$$\begin{aligned} \langle \{w\} | S_{\alpha}^+ S_{\beta}^+ | \{v\} \rangle &= \sum_{i=1}^{N_r} \frac{1}{v_i - \epsilon_{\beta}} \langle \{w\} | S_{\alpha}^+ | \{v\}_i \rangle \\ &\quad - \sum_{i' \neq i} \frac{1}{(v_i - \epsilon_{\beta})(v_{i'} - \epsilon_{\beta})} \langle \{w\} | S_{\alpha}^+ S_{\beta}^+ | \{v\}_{i,i'} \rangle. \end{aligned} \quad (24)$$

Here, the sets indicated by $\{v\}_i$ stand for sets where the rapidity i has been removed and similarly for $\{v\}_{i,i'}$, both i and i' rapidities have been removed. The $S^+ S^-$ form factor is given by¹³

$$\begin{aligned} \langle \{w\} | S_{\alpha}^- S_{\beta}^- | \{v\} \rangle &= \frac{\prod_{b=1}^{N_r} (w_b - \epsilon_{\alpha})(w_b - \epsilon_{\beta})}{\prod_{a=1}^{N_r-2} (v_a - \epsilon_{\alpha})(v_a - \epsilon_{\beta})} \frac{\det_{N_r} T(\alpha, \beta, \{w_b\}, \{v_a\})}{\prod_{b>a} (w_b - w_a) \prod_{b<a} (v_b - v_a)}, \end{aligned} \quad (25)$$

with

$$\begin{aligned} T_{ab}(\alpha, b) &= \prod_{c \neq a}^{N_r} (w_c - v_b) \left(\sum_{\gamma=1}^N \frac{1}{(v_b - \epsilon_{\gamma})(w_a - \epsilon_{\gamma})} \right. \\ &\quad \left. - 2 \sum_{c \neq a} \frac{1}{(v_b - w_c)(w_a - w_c)} \right), \quad b < N_r - 1, \end{aligned}$$

$$\begin{aligned} T_{aN_r-1}(\alpha, \beta) &= \frac{2w_a - \epsilon_{\alpha} - \epsilon_{\beta}}{[(w_a - \epsilon_{\alpha})(w_a - \epsilon_{\beta})]^2}, \\ T_{aN_r}(\alpha, \beta) &= \frac{1}{(w_a - \epsilon_{\alpha})^2}, \end{aligned} \quad (26)$$

where $\alpha \neq \beta$ is assumed, with the convention that it vanishes when $\alpha = \beta$. Note that $\langle \{w\} | S_{\alpha}^+ S_{\beta}^- | \{v\} \rangle$ is symmetric under the exchange of α and β , although this is not manifested in the formal expression. This nontrivial property will be checked during the numerical computation.

This correlation is then written as the sum of N_r^2 determinants, which is much less than the sum over the full Hilbert space needed in other approaches. In the following, we will determine a similar expression for $\langle S_{\alpha}^z S_{\beta}^z \rangle$ and then we will show that it is possible to reduce these formulas to sums of only N_r terms.

Determinant representation of $\langle S_{\alpha}^z S_{\beta}^z \rangle$

The operator $\mathcal{A}(u)$ only has simple poles at ϵ_{α} such that¹⁴

$$S_{\alpha}^z = \lim_{u \rightarrow \epsilon_{\alpha}} (u - \epsilon_{\alpha}) \mathcal{A}(u). \quad (27)$$

This allows one to write

$$\begin{aligned} \langle \{w\} | S_{\alpha}^z S_{\beta}^z | \{v\} \rangle &= \lim_{u \rightarrow \epsilon_{\beta}} \lim_{u' \rightarrow \epsilon_{\alpha}} (u' - \epsilon_{\alpha})(u - \epsilon_{\beta}) \\ &\quad \times \langle \{w\} | \mathcal{A}(u') \mathcal{A}(u) \prod_{i=1}^{N_r} \mathcal{B}(v_i) | 0 \rangle, \end{aligned} \quad (28)$$

which can be easily computed by commuting the \mathcal{A} operators until they reach the far right and act on the pseudo-vacuum in the following way:

$$\mathcal{A}(u) | 0 \rangle = a(u) | 0 \rangle = -\frac{1}{g} | 0 \rangle + \frac{1}{2} \sum_{\gamma=1}^N \frac{1}{u - \epsilon_{\gamma}} | 0 \rangle. \quad (29)$$

Using the commutation relation¹⁴

$$[\mathcal{A}(u), \mathcal{B}(v)] = \frac{\mathcal{B}(u)}{u - v} - \frac{\mathcal{B}(v)}{u - v} \quad (30)$$

and defining $\mathcal{G} \equiv \langle \{w\} | \mathcal{A}(u') \mathcal{A}(u) \prod_{i=1}^{N_r} \mathcal{B}(v_i) | 0 \rangle$, we find by commuting $\mathcal{A}(u)$ and $\mathcal{B}(v_1)$ that

$$\begin{aligned} \mathcal{G} &= \langle \{w\} | \mathcal{A}(u') \mathcal{B}(v_1) \mathcal{A}(u) \prod_{i=2}^{N_r} \mathcal{B}(v_i) | 0 \rangle \\ &\quad + \frac{\langle \{w\} | \mathcal{A}(u') | \{v\}_1; u \rangle}{u - v_1} - \frac{\langle \{w\} | \mathcal{A}(u') | \{v\} \rangle}{u - v_1}, \end{aligned} \quad (31)$$

where $|\{v\}_1; u\rangle$ is the (non-Bethe) state built by replacing the

rapidity v_1 by u . By commuting again $N_r - 1$ times and using Eq. (29), we find

$$\mathcal{G} = F(u) \langle \{w\} | \mathcal{A}(u') | \{v\} \rangle + \sum_{i=1}^{N_r} \frac{1}{u - v_i} \langle \{w\} | \mathcal{A}(u') | \{v\}_i; u \rangle, \quad (32)$$

where we defined $F(u) \equiv -\frac{1}{g} + \frac{1}{2} \sum_{\gamma=1}^{N_r} \frac{1}{u - \epsilon_\gamma} - \sum_{i=1}^{N_r} \frac{1}{u - v_i}$. The same procedure can then be repeated in order to have $\mathcal{A}(u')$ act on $|0\rangle$:

$$\begin{aligned} \mathcal{G} = F(u) & \left[F(u') \langle \{w\} | \{v\} \rangle + \sum_{i=1}^{N_r} \frac{\langle \{w\} | \{v\}_i; u' \rangle}{u' - v_i} \right] \\ & + \sum_{i=1}^{N_r} \frac{1}{u - v_i} \left[\sum_{i' \neq i}^{N_r} \frac{1}{u' - v_{i'}} \langle \{w\} | \{v\}_{i,i'}; u, u' \rangle + F_i(u') \right. \\ & \times \langle \{w\} | \{v\}_i; u \rangle + \frac{1}{u' - u} \langle \{w\} | \{v\}_i; u' \rangle \\ & \left. - \frac{1}{u' - u} \langle \{w\} | \{v\}_i; u \rangle \right], \quad (33) \end{aligned}$$

with $F_i(u) \equiv -\frac{1}{g} + \frac{1}{2} \sum_{\gamma=1}^{N_r} \frac{1}{u - \epsilon_\gamma} - \sum_{i' \neq i}^{N_r} \frac{1}{u - v_{i'}}$. It is then easy to take the limit as prescribed by Eq. (28) and find

$$\begin{aligned} \langle \{w\} | S_\alpha^\pm S_\beta^\pm | \{v\} \rangle &= \frac{\langle \{w\} | \{v\} \rangle}{4} + \sum_{i=1}^{N_r} \frac{\langle \{w\} | S_\alpha^\pm | \{v\}_i \rangle}{2(\epsilon_\alpha - v_i)} \\ &+ \sum_{i=1}^{N_r} \frac{\langle \{w\} | S_\beta^\pm | \{v\}_i \rangle}{2(\epsilon_\beta - v_i)} + \sum_{i' \neq i}^{N_r} \frac{\langle \{w\} | S_\alpha^\pm S_\beta^\pm | \{v\}_{i,i'} \rangle}{(\epsilon_\beta - v_i)(\epsilon_\alpha - v_{i'})}. \quad (34) \end{aligned}$$

Note that we use the same notation as in Eq. (21) for the S^\pm form factor but there, it is evaluated between two states with N_r and $N_r + 1$ rapidities, while here, it is between two states with $N_r - 1$ and N_r . This should not be a source of confusion.

The static correlation function can now be evaluated by setting $\{v\} = \{w\} = \{w_0\}$, the set of rapidities corresponding to the ground state of the system. Using Eqs. (21) and (25), we can directly express this correlation function as a sum of $N_r^2 + N_r$ determinants.

B. Reduction formulas

As anticipated in the Introduction, the previous expressions can be reduced to sums over only N_r determinants. This is explicitly worked out in the following. We will assume that $\alpha \neq \beta$ because for intralevel correlations from Eq. (6), we have

$$\langle \{w\} | S_\alpha^\pm S_\alpha^\pm | \{w\} \rangle = \frac{1}{2} + \langle \{w\} | S_\alpha^\pm | \{w\} \rangle, \quad (35)$$

which is already a single determinant expression.

1. Reduction of $\langle S_\alpha^\pm S_\beta^\pm \rangle$

Here, we need to evaluate Eq. (24) in the limit $v \rightarrow w$. In this case, Eqs. (21) and (25) for the form factors simplify to

$$\frac{\langle \{w\}_{N_r} | S_\alpha^\pm | \{w_q\}_{N_r-1} \rangle}{w_q - \epsilon_\beta} = \frac{w_q - \epsilon_\alpha}{w_q - \epsilon_\beta} \det_{N_r} U^{(q)}, \quad (36)$$

$$\frac{\langle \{w\}_{N_r} | S_\alpha^\pm S_\beta^\pm | \{w_q, l\}_{N_r-2} \rangle}{(w_q - \epsilon_\beta)(w_l - \epsilon_\beta)} = - \frac{(w_q - \epsilon_\alpha)(w_l - \epsilon_\alpha)}{w_l - w_q} \det_{N_r} U^{(q,l)}. \quad (37)$$

The matrices U are defined as follows. $U^{(q)}$ and $U^{(q,l)}$ are equal to the Gaudin matrix in Eq. (20) except for columns q and q, l , respectively, where

$$U_{aq}^{(q)} = U_{aq}^{(q,l)} = \frac{1}{(w_a - \epsilon_\alpha)^2}, \quad (38)$$

$$U_{al}^{(q,l)} = \frac{2w_a - \epsilon_\alpha - \epsilon_\beta}{(w_a - \epsilon_\alpha)^2(w_a - \epsilon_\beta)^2}. \quad (39)$$

In Eq. (37), it is explicitly assumed that $\alpha \neq \beta$ with the convention that for $\alpha = \beta$, it is zero. In fact, for $\alpha = \beta$, it is not difficult to prove that Eq. (36) reproduces the correct result given by Eq. (35).

Since everything is symmetric under exchange of l and q , we only perform the sum over $l < q$ and in the end, we multiply the result by 2. Thus, we need to perform the sum (neglecting for the moment the l independent factors)

$$\sum_{l=1}^{q-1} \frac{w_l - \epsilon_\alpha}{w_l - w_q} \det U^{(q,l)} \equiv \sum_{l=1}^{q-1} K_{lq} \det U^{(q,l)}. \quad (40)$$

Let us write the matrix $U^{(q,l)}$ as a vector of vectors,

$$U^{(q,l)} = |\vec{G}_1, \dots, \vec{G}_{l-1}, \vec{B}, \vec{G}_{l+1}, \dots, \vec{G}_{q-1}, \vec{C}, \vec{G}_{q+1}, \dots, \vec{G}_{N_r}|, \quad (41)$$

where the \vec{G}_i corresponds to the columns of the matrix G , \vec{C} (at position q) corresponds to the vector given by Eq. (38), and \vec{B} corresponds to the vector given by Eq. (39).

The sum we want to calculate is (we use $|\cdot|$ for the determinant)

$$\begin{aligned} \sum_{l=1}^{q-1} K_{lq} \det U^{(q,l)} &= K_{1q} |\vec{B}, \vec{G}_2, \vec{G}_3, \dots| + K_{2q} |\vec{G}_1, \vec{B}, \vec{G}_3, \dots| \\ &+ K_{3q} |\vec{G}_1, \vec{G}_2, \vec{B}, \dots| + \dots \end{aligned} \quad (42)$$

Using the fact that two determinants which differ by a single column can be easily expressed as a single determinant, the first two terms of the sum be written as $|K_{2q} \vec{G}_1 - K_{1q} \vec{G}_2, \vec{B}, \vec{G}_3, \dots|$. Elementary column operations allow us to write the third determinant as $K_{3q} |\vec{G}_1, \vec{G}_2, \vec{B}, \dots| = K_{3q} |\vec{G}_1 - \frac{K_{1q}}{K_{2q}} \vec{G}_2, \vec{G}_2, \vec{B}, \dots|$. This term differs by a single column from the preceding sum. We can then write the sum of the first three terms as a single determinant, $K_{3q} |\vec{G}_1 - \frac{K_{1q}}{K_{2q}} \vec{G}_2, \vec{G}_2 - \frac{K_{2q}}{K_{3q}} \vec{G}_3, \vec{B}, \dots|$. We can keep on adding terms in the same way until we reach column $q-1$ and find that Eq. (42) can be written as a single determinant,

$$K_{q-1q} \left| \tilde{G}_1 - \frac{K_{1q}}{K_{2q}} \tilde{G}_2, \tilde{G}_2 - \frac{K_{2q}}{K_{3q}} \tilde{G}_3, \dots, \tilde{B}, \tilde{C}, \tilde{G}_{q+1}, \dots, \tilde{G}_{N_r} \right|. \quad (43)$$

In this way, we reduced the double sum to a single one. The additional terms in the correlation function (coming from $\langle S^- \rangle$) can be also incorporated to the sum in a similar fashion. The $\langle S^- \rangle$ term in Eq. (24) is given by Eq. (36) and can be simply encoded in the representation we just obtained for the sum over $l < q$ of $\langle S^- S^- \rangle$. In this way, we finally have the following for the full correlation function:

$$\langle \{w\} | S_\alpha^- S_\beta^+ | \{w\} \rangle = \sum_{q=1}^{N_r} \frac{w_q - \epsilon_\alpha}{w_q - \epsilon_\beta} \det D_q^{(\alpha, \beta)}, \quad (44)$$

where we defined the matrix

$$D_q^{(\alpha, \beta)} = [\tilde{D}_{q,1}^{(\alpha, \beta)}, \tilde{D}_{q,2}^{(\alpha, \beta)}, \dots, \tilde{D}_{q,N_r}^{(\alpha, \beta)}] \quad (45)$$

that has the following structure:

$$D_{q,i}^{(\alpha, \beta)} = \begin{cases} \tilde{G}_i - \frac{K_{iq}}{K_{i+1q}} \tilde{G}_{i+1}, & i < q-1 \\ \tilde{G}_i + 2 \frac{(w_q - \epsilon_\beta)(w_{q-1} - \epsilon_\alpha)}{w_{q-1} - w_q} \tilde{B}, & i = q-1 \\ \tilde{C}, & i = q \\ \tilde{G}_i, & i > q. \end{cases} \quad (46)$$

The low level of complexity of this representation as sum of N_r determinants of N_r by N_r matrices allows us to easily access the static correlation functions for systems with a large number of pairs compared to previously published results.

We stress again that these formulas are true only for $\alpha \neq \beta$.

2. Reduction of $\langle S_\alpha^z S_\beta^z \rangle$

According to Eq. (34), when $v \rightarrow w$,

$$\begin{aligned} \langle \{w\} | S_\alpha^z S_\beta^z | \{w\} \rangle &= \frac{1}{4} \langle \{w\} | \{w\} \rangle - \frac{1}{2} \left[\sum_{i=1}^{N_r} \frac{\langle \{w\} | S_\beta^- | \{w\}_i \rangle}{w_i - \epsilon_\beta} \right. \\ &\quad \left. - \sum_{i' \neq i}^{N_r} \frac{\langle \{w\} | S_\alpha^- S_\beta^- | \{w\}_{i,i'} \rangle}{(w_i - \epsilon_\beta)(w_{i'} - \epsilon_\alpha)} \right] \\ &\quad - \frac{1}{2} \left[\sum_{i=1}^{N_r} \frac{\langle \{w\} | S_\alpha^- | \{w\}_i \rangle}{w_i - \epsilon_\alpha} \right. \\ &\quad \left. - \sum_{i' \neq i}^{N_r} \frac{\langle \{w\} | S_\beta^- S_\alpha^- | \{w\}_{i,i'} \rangle}{(w_i - \epsilon_\alpha)(w_{i'} - \epsilon_\beta)} \right]. \quad (47) \end{aligned}$$

From Eqs. (44) and (45), it is straightforward to show that when $\alpha \neq \beta$,

$$\langle \{w\} | S_\alpha^z S_\beta^z | \{w\} \rangle = \frac{\|w\|^2}{4} - \frac{1}{2} \sum_{q=1}^{N_r} (\det D_q^{(\alpha, \beta)} + \det D_q^{(\beta, \alpha)}). \quad (48)$$

For $\alpha = \beta$, the result is trivially $\langle \{w\} | (S_\alpha^z)^2 | \{w\} \rangle = 1/4$.

This completes the representation of the static correlation functions in terms of determinants. To make further progress, we need explicit results for the ground-state rapidities $\{w\}$, i.e., the lowest-energy solutions to the Richardson equations. The following section is devoted to this.

IV. SOLUTION OF THE RICHARDSON EQUATIONS FOR THE GROUND STATE

A. General properties

At $g=0$, for N_r pair vacancies in N energy levels ϵ_α with only double degeneracy, the $\binom{N}{N_r}$ solutions to the Richardson equations are trivial. They are given by Eq. (12) with the N_r rapidities set to be strictly equal to one of the energies ϵ_α . Clearly, the GS in that limit is built by choosing the N_r highest-energy levels, i.e., $w_1 = \epsilon_N, w_2 = \epsilon_{N-1}, \dots, w_{N_r} = \epsilon_{N-N_r+1}$.

Apart from a few particular cases with a small number of particles, the Richardson equations are not solvable analytically when $g \neq 0$. The perturbative expansions for small^{23,24} and large^{25,26} couplings are not predictive for all values of the pairing strength and so the most accurate results come from the numerical solution. The solutions are such that every w_j is either a real quantity or forms, with another parameter $w_{j'}$, a complex conjugate pair (CCP), i.e. $w_{j'}^* = w_j$. The mechanism for the CCP formation is very easy: as interactions are turned on, all w_j are real quantities for small enough g , but at a certain critical value of the coupling g_j^* , two rapidities will be exactly equal to one of the energy levels $[w_j = w_{j'} = \epsilon_\gamma(j)]$ and for $g > g_j^*$, the two parameters that collapsed will form a CCP at least for a finite interval in g . The situation is in fact rather intricate: the values g_j^* are implicit functions of all other rapidities and can only be read off a full solution of the Richardson equations for a specific choice of state. Moreover, CCPs can split back into real pairs, whose components can then re-pair with neighboring rapidities. Finding complex solutions to the Richardson equations is thus difficult in general, since there is no equivalent to the “string hypothesis” as for, e.g., integrable spin chains.

The solutions for the ground state have a particularly simple structure. In fact, the set of critical points is such that the smaller a rapidity is at $g=0$, the smaller the g at which it forms a CCP will be. As we raise g from zero, there will come a point at which w_{N_r} will form a CCP with w_{N_r-1} when they are both equal to ϵ_{N-N_r+2} . As g is raised some more, w_{N_r-2} and w_{N_r-3} will form a CCP at ϵ_{N-N_r+4} and this will go on until every rapidity has formed a CCP in the case of even N . Oppositely, with an odd number of rapidities in the system, w_1 (the largest one at $g=0$) will always remain a real quantity no matter how large the coupling strength is. After the CCPs are formed, no further collapse happens in the case of the ground state, while for excited states, further collapses

can take place and complex solutions can become real again.

Different choices of the parameters ϵ_α and of their eventual degenerations specify different models. In all the preceding sections, everything was completely general (modulo having to take some extra precautions in the case of coinciding levels ϵ_α), but from now on, we specialize to the case of equally spaced doubly degenerate levels. We make the choice to use $\epsilon_\alpha = \alpha$ which sets the zero of energy and implies that every energy will be given in units of the (pair) inter-level spacing. Furthermore, we consider only half-filling of the energy levels ($N = 2N_r = 2N_p = N_f$). In this case, as $g \rightarrow \infty$, the real part of every rapidity will go to $+\infty$, whereas the CCP imaginary parts will go to $\pm\infty$.

B. Numerical procedure and results

At the precise value of g at which a pair of rapidities $(w_j, w_{j'})$ collapse into a CCP [$w_j = w_{j'} = \epsilon_\gamma(j)$], the Richardson equations [Eq. (14)] labeled j and j' will include two diverging terms whose sum remains finite. In order to be able to treat these points numerically, one can define the following real variables:

$$w_{1,j} \equiv w_j + w_{j'} \quad (49)$$

$$w_{2,j} \equiv \frac{2\epsilon_\gamma(j) - w_j - w_{j'}}{(w_j - w_{j'})^2}, \quad (50)$$

whose inverse transformation reads

$$w_j = \frac{1}{2} \left[w_{1,j} + \sqrt{\frac{2\epsilon_{j-1} - w_{1,j}}{w_{2,j}}} \right], \quad (51)$$

$$w_{j'} = \frac{1}{2} \left[w_{1,j} - \sqrt{\frac{2\epsilon_{j-1} - w_{1,j}}{w_{2,j}}} \right]. \quad (52)$$

As discussed in Ref. 23, we need to know beforehand which rapidities will form a CCP and at which $\epsilon_\gamma(j)$ it will happen in order to use this type of change of variables. Since we only need ground-state solutions in this paper, this requirement is easily met.

At the critical point, $w_{2,j}$ goes to a well defined (though *a priori* unknown) finite 0/0 form. Using it as a variable in the system of equations therefore avoids some potential numerical complications when close to a critical point.

By multiplying the j and j' Richardson equations by, respectively, $\epsilon_\gamma(j) - w_j$ and $\epsilon_\gamma(j) - w_{j'}$, we can get rid of the divergences that show up at critical points. Adding the resulting equations (giving $F_{1,j}$) and subtracting them and then dividing it by $w_j - w_{j'}$ (giving $F_{2,j}$), it is simple to obtain the following two real equations:

$$F_{1,j} = \sum_{i \neq j-1}^N \frac{(\epsilon_{j-1} - \epsilon_i)(2\epsilon_i - w_{1,j})}{(\epsilon_i)^2 - \left(\epsilon_i - \frac{w_{1,j}}{4}\right)w_{1,j} - \frac{2\epsilon_{j-1} - w_{1,j}}{4w_{2,j}}} - \sum_{j' \neq j, j-1}^{N_r} 2 \frac{(\epsilon_{j-1} - w_{j'}) (2w_{j'} - w_{1,j})}{(w_{j'})^2 - \left(w_{j'} - \frac{w_{1,j}}{4}\right)w_{1,j} - \frac{2\epsilon_{j-1} - w_{1,j}}{4w_{2,j}}} + \frac{2\epsilon_{j-1} - w_{1,j}}{g} + 2(N-1) - 4(N_r-2) = 0, \quad (53)$$

$$F_{2,j} = \sum_{i \neq j-1}^N \frac{(\epsilon_{j-1} - \epsilon_i)}{(\epsilon_i)^2 - \left(\epsilon_i - \frac{w_{1,j}}{4}\right)w_{1,j} - \frac{2\epsilon_{j-1} - w_{1,j}}{4w_{2,j}}} - \sum_{j' \neq j, j-1}^{N_r} 2 \frac{(\epsilon_{j-1} - w_{j'})}{(w_{j'})^2 - \left(w_{j'} - \frac{w_{1,j}}{4}\right)w_{1,j} - \frac{2\epsilon_{j-1} - w_{1,j}}{4w_{2,j}}} - \frac{1}{g} + 2w_{2,j} = 0. \quad (54)$$

The resulting system of nonlinear equations can then be easily solved using Newton's method. Notice that every element of the Jacobian matrix has an analytical expression that is easy to obtain, and therefore is not explicitly written here. Of course, for Newton's procedure to converge to the correct solution at a given g , we need a good approximation to it. It is simple to do so by slowly incrementing g starting from $g=0$, where the GS is known. One can then use a simple linear regression on $w_{1,j}, w_{2,j}$ to obtain an educated guess to the ground state at $g' = g + \Delta g$. Despite its simplicity, this method, very similar to other ones in the literature,^{6,23,27,28} is sufficient for obtaining the ground-state solutions. For general states, for which the formation of and splitting apart of CCPs can be highly nontrivial, a more refined algorithm (see Refs. 29 and 30 for example) is needed to find the solutions.

Figure 1 shows three examples of the numerically computed ground-state solution of Richardson's equations. One can see that the generic statements made about this solution in the preceding subsection are confirmed. As g gets sufficiently large and every rapidity has collapsed into a CCP (for an even number of pairs), they arrange themselves into an arc in the complex plane, as shown more clearly in Fig. 2. This behavior was originally predicted using the analogy between the set of equations and a two dimensional electrostatic problem involving fixed and free charges.^{18,22,31}

For a correct interpretation of the main features of the solutions to the Richardson equations, it is important to know the value of the superconducting gap given by Eq. (15) for the particular Hamiltonian we choose (i.e., $\epsilon_\alpha = \alpha$). It is easy to show that for large N ,

$$\Delta = \frac{\Delta_{GC}}{N} = \frac{1}{2 \sinh 1/2g}, \quad (55)$$

an expression we will need to compare finite-size results with the grand-canonical ones. Consequently, Anderson's

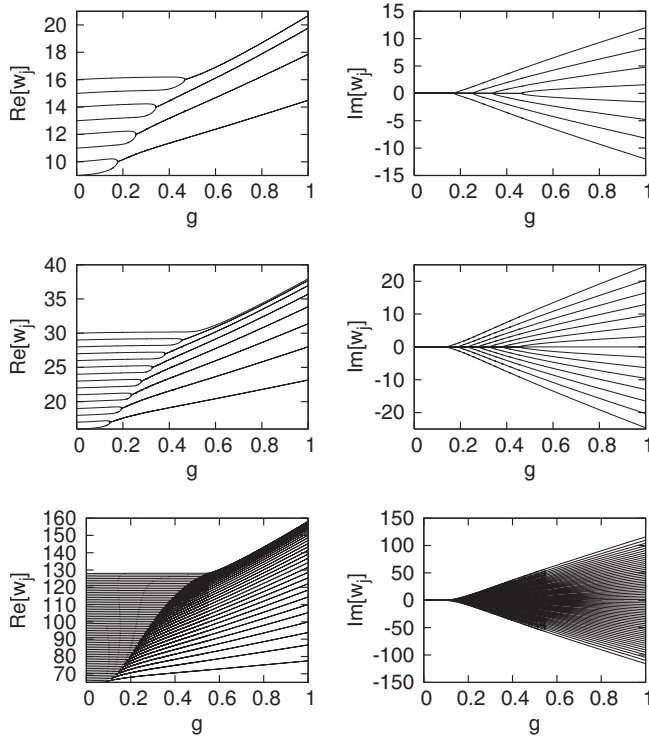


FIG. 1. g dependence of the real (left) and imaginary (right) parts of the ground-state rapidities. From top to bottom, they correspond to $N_r=8, 15, 64$ always at half-filling ($N=2N_r$).

criterion⁷ for the presence of superconductivity for large N is

$$\Delta \gtrsim N \Rightarrow g \gtrsim \frac{1}{2 \ln 2N_r}, \quad (56)$$

showing the typical²⁴ logarithmic behavior of the small g expansion.

Figure 3 shows, as a function of N_r , the values of the coupling constant $g_{N_r}^*(N_r)$ at which the first two rapidities form a CCP. We also plot, for even N_r , the values of $g = g_1^*(N_r)$ at which the last couple of rapidities collapse into a

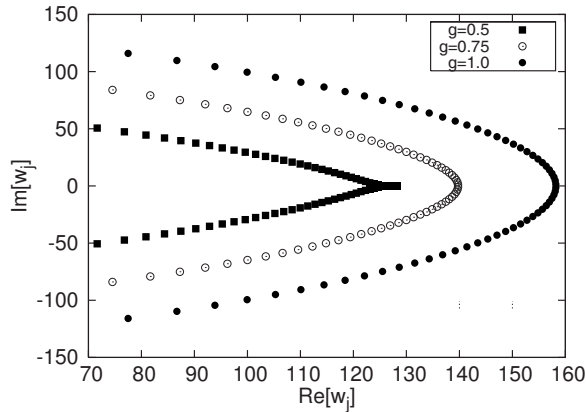


FIG. 2. Location in the complex plane of the ground-state rapidities showing the formation of the arclike solution. All the values correspond to $N=128, N_r=64$.

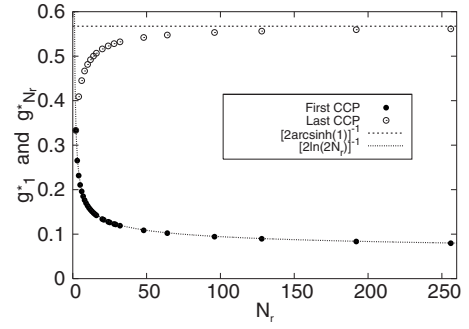


FIG. 3. Values of the coupling parameter at which the first and the last CCPs are formed.

CCP. The latter is limited at large N_r by³¹ $g_0 = (2 \operatorname{arcsinh} 1)^{-1} = 0.567\,296$, a constant which is also shown in the figure.

These two numbers are particularly relevant to understand qualitatively the different behaviors as a function of g and N_r . In fact, when g is larger than $g_1^*(N_r)$, all the particles are paired and the system has entered its asymptotic superconducting regime. Oppositely, when no pair has still been formed, i.e., for $g < g_{N_r}^*(N_r)$, superconductivity is absent. In fact, $g_{N_r}^*(N_r)$ coincides with the critical value of the coupling given by Anderson's criterion [Eq. (56)] for large N . The curve resulting from Eq. (56) is plotted in Fig. 3 and the agreement with $g_{N_r}^*(N_r)$ is excellent even for a relatively small value of N_r .

Note also that $g_{N_r}^*(N_r)$ vanishes in the thermodynamic limit, which can be simply interpreted as the Cooper instability. A quantitative understanding of these phenomena and of the crossover between small and large g at fixed finite N_r requires an accurate study of the correlation functions, which we will present in the next section.

C. Ground-state energy

In Fig. 4, we plot the value of the ground-state energy per pair (in units of the inter-level spacing) at half-filling for a set of different numbers of pairs as given by $N_p E_0 = \sum_{j=1}^N \frac{\epsilon_j}{2} - \sum_j w_j$.

One interesting feature is the presence of a size invariant point at which every curve crosses (see the inset of Fig. 4 for

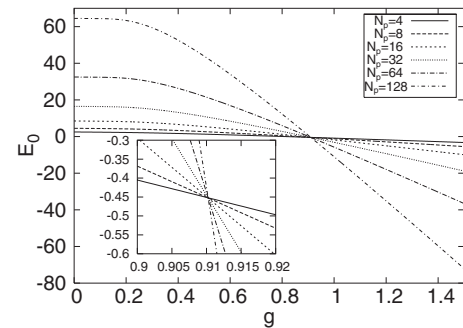


FIG. 4. Ground-state energy per pair E_0 as a function of g . Inset: Zoom close to the crossing point.

a zoom close to this point). Indeed, at $g_{inv} \approx 0.910$, the ground-state energy seems to be independent of the number of pairs in the system, $E(g_{inv}) \approx -0.45$. However, the presence of this “fixed point” does not carry any deep meaning and can be easily understood in terms of the $1/N_p$ expansion developed in Refs. 22 and 32. In fact, according to these references, for large N_p , the ground-state energy per particle can be written as

$$E_0 = N_p E_0^{(0)} + E_0^{(1)} + O(1/N_p), \quad (57)$$

with $[E_0^{(0)}]$ is nothing but Eq. (16)

$$E_0^{(0)} = 1 - \frac{1}{2} \coth \frac{1}{2g}, \quad (58)$$

$$E_0^{(1)} = \frac{1}{2} [1 - \phi(2g) \coth 1/(2g)], \quad (59)$$

$$\phi(2g) = \frac{2}{\pi} \int_0^\infty \frac{dx}{1+x^2} \frac{\cosh \pi x/2}{\sqrt{\cosh^2(\pi x/2) + \sinh^2[1/(2g)]}},$$

where we adapt the results to our normalization (i.e., the quantities of Ref. 32 reads $D=2N_p$, $\lambda=2g$ and there is a global shift of the energy levels). The scale invariant point just corresponds to the value of g for which the order N_p term $E_0^{(0)}$ vanishes, i.e., $g_{inv} = (2 \operatorname{arccoth} 2)^{-1} = 0.910\,239 \dots$. The energy at this point, apart from $O(1/N_p)$ corrections, is independent of N_p and given by $E_0^{(1)}(g_{inv}) = -0.452\,76 \dots$. Equation (57) is thus practically a perfect approximation of the actual value of the ground-state energy for large enough N_p , say, $N_p \geq 16$.

V. CALCULATION OF THE CORRELATION FUNCTIONS

The formulas we obtained for the correlation functions are completely general and are valid for any choice of the Hamiltonian parameters ϵ_α and g (some care would have to be taken in the limit of coinciding energy levels, however). To obtain a physical result, we still have to perform the sum over the N_f terms, introducing the solution to the Richardson equations in the determinants for the form factors. This cannot be done analytically, so we need to make a choice of the model to study. As we already mentioned, we only consider the most-studied case in the condensed matter literature, which consists of N equidistant levels at half-filling, i.e., $N = 2N_r = 2N_p = N_f$. We normalize the levels as

$$\epsilon_\alpha = \alpha \quad \text{with } \alpha = 1, \dots, N, \quad (60)$$

i.e., we measure the energy scale in terms of the interlevel spacing and we fix the Debye frequency (the largest energy level) to N .

A. Correlations among the same level and canonical order parameter

Among the various correlation functions, a central role is played by the ones on the same level. We consider the correlation

$$u_\alpha v_\alpha = \sqrt{\langle S_\alpha^- S_\alpha^+ \rangle \langle S_\alpha^+ S_\alpha^- \rangle} = \sqrt{1/4 - \langle S_\alpha^z \rangle^2}, \quad (61)$$

which can be easily obtained by the previous representation of $\langle S_\alpha^z \rangle$ and does not require the reduction formulas because it is written in terms of a single form factor. This correlation is important because it is one of the building blocks of the BCS theory and because it allows to define a “canonical” BCS order parameter. In fact, as already discussed, Eq. (8), defining the grand-canonical gap, is always zero in the canonical ensemble. Thus, following Ref. 33, we use as a canonical order parameter,

$$\Psi = \sum_{\alpha=1}^N u_\alpha v_\alpha. \quad (62)$$

Note that Ψ is just half of the concurrence (which is a local entanglement measure, see Ref. 34 as a review) which has been already calculated with the present method.¹⁶

In the large N limit, all these correlators must reduce to the value in the grand-canonical ensemble, which, from Eq. (9) specialized to $\epsilon_\alpha = \alpha$, is

$$u_\alpha v_\alpha = \frac{1}{2} \frac{\Delta}{\sqrt{\Delta^2 + (\alpha - N_p)^2/N^2}}, \quad (63)$$

where we fixed the chemical potential to $\mu = N/2 = N_p$ and we recall that Δ is given by Eq. (55). Consequently, in the same limit, the canonical order parameter is

$$\begin{aligned} \Psi_{N_p \rightarrow \infty} &= \lim_{N_p \rightarrow \infty} \sum_{\alpha=1}^{2N_p} u_\alpha v_\alpha = \frac{N\Delta}{4} \int_{-1}^1 \frac{dx}{\sqrt{\Delta^2 + (x/2)^2}} \\ &= \frac{N\Delta}{2} \log \frac{\sqrt{1+4\Delta^2}+1}{\sqrt{1+4\Delta^2}-1} = \frac{N\Delta}{2g} = \frac{N}{4g \sinh 1/2g}. \end{aligned} \quad (64)$$

It is evident that Ψ vanishes when the gap Δ is zero, confirming that in the thermodynamic limit, it is a good order parameter.

Our results for $u_\alpha v_\alpha$ are reported in Figs. 5 and 6. In the former, each plot consists of the various curves at fixed $g(=0.1, 0.2, 0.4, 0.7)$ with varying N_p . The latter instead shows the g dependence at fixed N_p . In Fig. 5, also the BCS results for any g are reported for comparison. It is evident that for all g , the results tend to converge to the BCS ones, as they must. However, this convergence is slower as g is smaller; for example, for $g=0.1$, the maximum at $N=256$ is only 90% of the asymptotic result and, conversely, at $g=0.7$, the $N_p=16$ result is already 99.8%. These finite N_p correlations are symmetric with respect to $(N+1)/2$ by construction. However, we point out that this will not be true for different level correlations, while in the grand-canonical ensemble, they are both symmetric.

In Fig. 7, we report the order parameter Ψ/N as a function of g for several values of N_p and compare it with the BCS result. This figure is exactly the same as the one for the concurrence obtained by Dunning *et al.*,¹⁶ with the important difference that they considered only $N_p \leq 34$ while we pushed the calculation up to $N_p = 128$. We could have calculated these correlations for larger N_p , but the ones considered

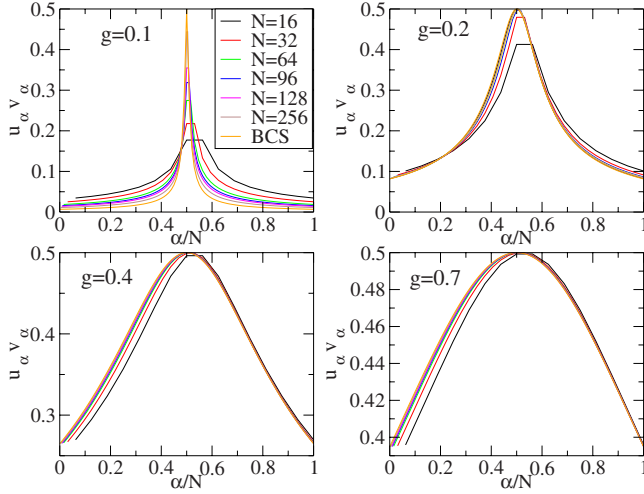


FIG. 5. (Color online) $u_\alpha v_\alpha$ as a function of α . Each plot is at fixed g and for several $N_p = N/2$.

are already enough to describe the crossover from the mesoscopic to the macroscopic regime. In fact, Fig. 7 shows that for $N_p = 128$, Ψ is almost indistinguishable from the BCS one [Eq. (64)], except for very small g that are characterized by the scaling in Eq. (56).

From the figure, it is evident that for $g < g^* \sim 0.18$, the BCS limit is reached from above, whereas for $g > g^*$, it is approached from below. Exactly at $g = g^*$, all the curves seem to cross in the same point. This is slightly different from what was observed before for a small number of particles,^{11,17} where, to get a similar crossing, the order parameter was multiplied by $N^{-\eta}$ with $\eta \approx 0.94$. To clarify this point, we zoom in on the crossing point in Fig. 8. It is evident that for $N_p \geq 8$, all the curves approximately cross in g^* , but this is not the case for smaller sizes. It is then direct to interpret g^* as the value of g where the leading finite-size correction of order $1/N$ vanishes (in fact, these are clearly

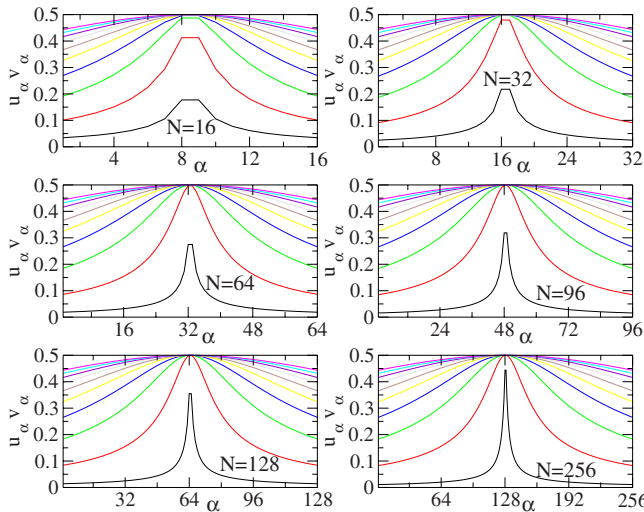


FIG. 6. (Color online) $u_\alpha v_\alpha$ as a function of α . Each plot is at fixed N_p but for several different g going from 0.1 (always the smallest) to 1 (always the largest) increasing by steps of 0.1.

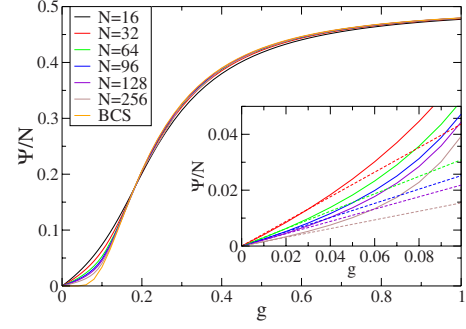


FIG. 7. (Color online) Canonical order parameter Ψ as a function of g for different numbers of pairs $N_p = N/2$. Inset: Small g behavior of Ψ compared with the large N expansion (dashed lines).

negative for large g and positive for very small ones). The differences for smaller size are due to higher order corrections $\sim 1/N^2$. This fixed point is thus completely analogous to results discussed in the previous section for the ground-state energy. A very interesting problem would be to calculate g^* directly from the finite-size form in an analytical manner using the $1/N_p$ expansion previously discussed.^{22,32}

The finite-size scaling $\Psi \sim N^\eta$ found in Refs. 11 and 17 can be clearly not true for large sizes, since Ψ is an extensive quantity. To check for which sizes it stops working, in the inset of Fig. 8, we plot ΨN^η . All the systems with sizes $N_p \leq 16$ cross indeed at the value of Refs. 11 and 17, $g_{cr} \sim 0.157$, but larger systems clearly deviate from this fixed point. We can then safely conclude that this scaling ansatz is effective only for $N_p \leq 16$. In Ref. 11, a second crossing point has been also found for a larger value of the pairing constant. According to our analysis also, this fixed point is present only for a relatively small number of pairs.

For large g , the BCS result is the leading term for large N and easily gives $\Psi/N = 1/2 - 1/(48g^2) + O(g^{-3}, N^{-1})$, whereas for small coupling, we have²⁴

$$\frac{\Psi}{N} = g \frac{\ln(3 + \sqrt{8})}{\sqrt{N_p}} + O(1/\ln N_p) \quad \text{for } g \ll 1. \quad (65)$$

Note that for large g , we have an N_p independent result, while for small g , there is a square-root singularity in N_p . The latter is again a manifestation of the nonperturbative

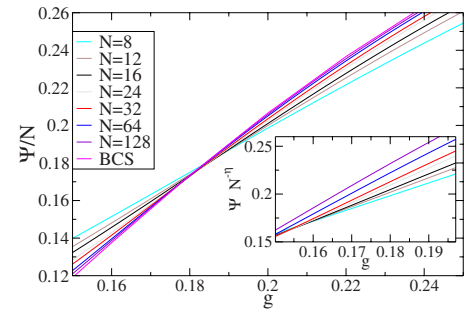


FIG. 8. (Color online) Zoom of Ψ in the region $0.15 < g < 0.25$ for several $N_p = N/2$. Inset: Scaling ansatz of Refs. 11 and 17 and its failure for large N_p .

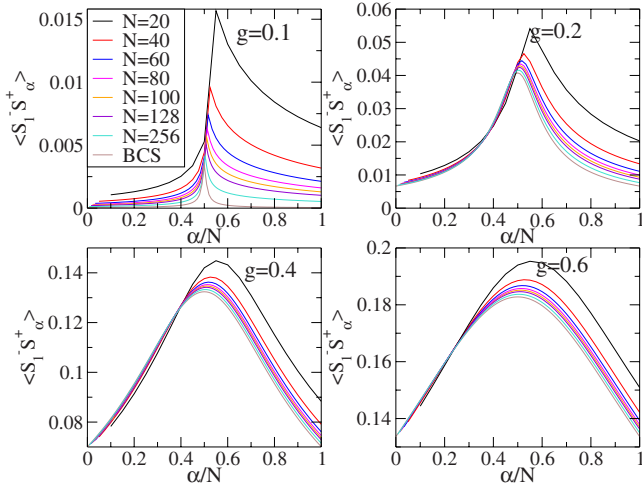


FIG. 9. (Color online) Off-diagonal correlations $\langle S_1^- S_\alpha^+ \rangle$ as a function of α/N .

nature of superconductivity. Both these analytical results are perfectly reproduced by our numerics.

B. Static correlation functions among different levels

Despite several interesting features of the correlation functions among the same levels that we have just discussed, these are qualitatively very similar to the grand-canonical ones. On the other hand, correlation functions between different levels (known as off-diagonal ones) are a strong signature of the canonical BCS-like pairing correlations and should be relevant for the interpretation of tunneling experiments. In fact, within the grand-canonical ensemble (and so for $N \rightarrow \infty$), these four-point correlation functions factorize to the product of two-point ones (i.e., in this approximation, the Cooper pairs are free). Oppositely, in the canonical ensemble, they are nontrivial functions of both the pairs as a consequence of quantum fluctuations. Following Ref. 11, we concentrate here on the two correlation functions

$$\langle S_1^- S_\alpha^+ \rangle \quad \text{and} \quad \langle S_1^z S_\alpha^z \rangle. \quad (66)$$

Our results for different values of g and N are reported in Fig. 9 and 10, respectively.

For $N \rightarrow \infty$, as a consequence of factorization, we have

$$\begin{aligned} \langle S_\alpha^- S_\beta^+ \rangle &= u_\alpha v_\alpha u_\beta v_\beta \\ &= \frac{1}{4} \frac{\Delta}{\sqrt{\Delta^2 + (\alpha - N_p)^2/N^2}} \frac{\Delta}{\sqrt{\Delta^2 + (\beta - N_p)^2/N^2}}. \end{aligned} \quad (67)$$

In particular, at fixed β , these correlations are symmetric with respect to $\alpha = N_p$. Including some trivial finite-size effect, in the grand-canonical ensemble, one would expect a symmetry at $(N+1)/2$ as for the on-level correlations. However, as evident from the figures, this is not the case in the canonical description. Furthermore, the smaller g is, the more asymmetrical are the correlations. Such asymmetries are very pronounced for all the $S^- S^+$ correlators, as, for ex-

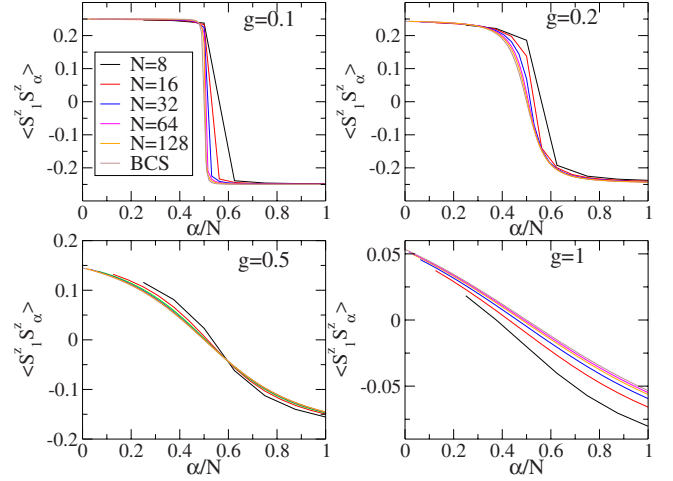


FIG. 10. (Color online) $\langle S_1^z S_\alpha^z \rangle$ as a function of α/N .

ample, shown in Fig. 11, where we report the correlator $\langle S_{N_p+1}^- S_\alpha^+ \rangle$ as the other extreme (compared to $\langle S_1^- S_\alpha^+ \rangle$). Thus, the asymmetries can be used to understand the degree of “canonicity” of a system. Note, in particular, the very different scales in Figs. 9 and 11: off-diagonal correlation functions are much more important when one of the levels is close to the Fermi point, a fact that is not surprising, being true also in the grand-canonical ensemble.

A last property that is not apparent from the plots but that is true (even if not evident from the determinant representation) is that

$$\langle S_\alpha^- S_\beta^+ \rangle = \langle S_\beta^- S_\alpha^+ \rangle, \quad (68)$$

which we checked for all the values we calculated.

The correlation function $S_\alpha^z S_\beta^z$ in the grand-canonical ensemble also factorizes into the product of two-point ones:

$$\langle S_\alpha^z S_\beta^z \rangle = \frac{(\alpha - N_p)/2N}{\sqrt{\Delta^2 + (\alpha - N_p)^2/N^2}} \frac{(\beta - N_p)/2N}{\sqrt{\Delta^2 + (\beta - N_p)^2/N^2}}, \quad (69)$$

and it is an odd function at $\alpha = M$ (or β). Again, the finite N results do not have this symmetry, which is recovered only in

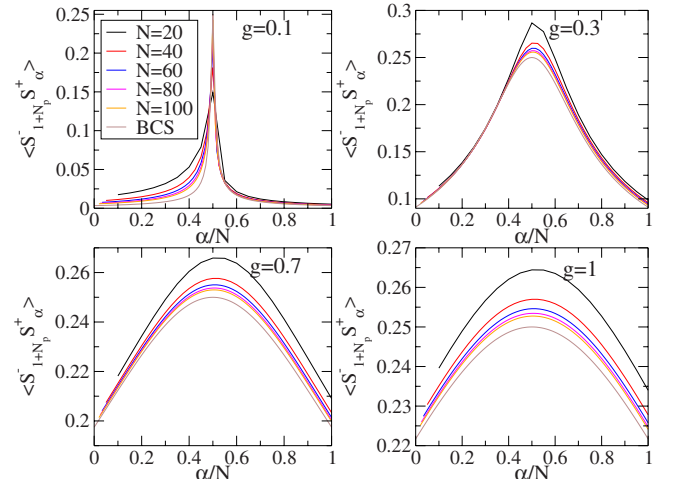


FIG. 11. (Color online) $\langle S_{N_p+1}^- S_\alpha^+ \rangle$ as a function of α/N .

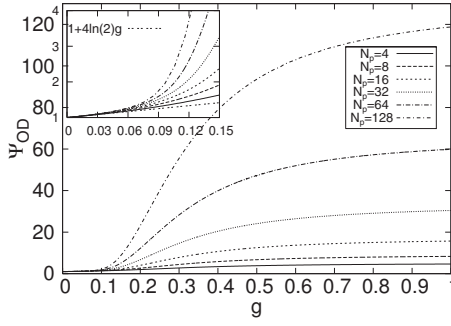


FIG. 12. Summed correlation function $\frac{1}{N_p} \sum_{\alpha, \beta=1}^N \langle S_{\alpha}^+ S_{\beta}^+ \rangle$ as a function of g for various numbers of pairs. Inset: small g behavior compared with the analytic expression.

the thermodynamic limit. The smoothing of the steplike structure increases g , a well-known effect also in the grand-canonical ensemble.

C. Off-diagonal order parameter

Another fundamental quantity is the so-called off-diagonal long-range order parameter³⁵ defined by

$$\Psi_{OD} \equiv \frac{1}{N_p} \sum_{\alpha, \beta=1}^N \langle S_{\alpha}^+ S_{\beta}^- \rangle, \quad (70)$$

which, as a difference with Ψ , takes into account the effect of nondiagonal correlations. Ψ_{OD} is clearly accessible from the direct computation of the off-diagonal correlation functions (as already proposed¹³), but this is not needed. In fact, it can be obtained without the use of the determinant representation using the Hellmann-Feynman theorem (an alternative method of calculation has been also proposed³⁶). The derivative of the ground-state energy with respect to the coupling strength allows us to directly compute the double sum over all levels of the static $S^+ S^-$ correlation function, i.e.,

$$\Psi_{OD} = \frac{1}{N_p} \sum_{\alpha, \beta=1}^N \langle S_{\alpha}^+ S_{\beta}^- \rangle = - \frac{1}{N_p} \frac{\partial E_0(g)}{\partial g}. \quad (71)$$

Figure 12 shows this summed correlation for different pair numbers. At $g=0$, the only contributing terms are the one for which $\beta=\alpha$ since no correlations between pairs in different levels exist and we trivially have $\sum_{\alpha, \beta=1}^N \langle S_{\alpha}^+ S_{\beta}^- \rangle = N_p$. As the interaction is turned on, interlevel correlations build up rapidly until they saturate for maximally correlated wave functions. For every N_p , this large g limit is clearly given by $\Psi_{OD} = N_p + 1$.

The small g behavior can be obtained analytically from the known result for the energy:²⁴

$$1 - E_0 = g + 2g^2 \ln 2 + O[g^3, (\ln L)^{-1}] \\ \Rightarrow \Psi_{OD} = 1 + g4 \ln 2 + O(g^2). \quad (72)$$

This curve is shown in the inset of Fig. 12 and perfectly agrees with the numerical results. Again, the deviations from this behavior start to occur at a value of g given by the usual logarithmic scaling of Eq. (56).

In the thermodynamic limit, as a consequence of the factorization, Ψ_{OD} is trivially related to Ψ as

$$\Psi_{OD}^{N_p \rightarrow \infty} = \frac{\Psi_{N_p \rightarrow \infty}^2}{N_p}, \quad (73)$$

signaling that one is extensive if and only if the other one is. However, as evident from the figure, for fixed finite N_p , this is not true and the two quantities are independent. Furthermore, for small g , in the regime that is not accessible by the BCS ansatz, they are both linear in g and cannot be in a quadratic relation as for large N_p . Actually, it has been proven³⁷ that Ψ and Ψ_{OD} satisfy the following relations for any value of g and N_p :

$$\frac{1}{N_p} \Psi(\Psi - 1) \leq \Psi_{OD} \leq 1 + \frac{N}{N_p} \Psi, \quad (74)$$

which for $N_p \rightarrow \infty$ are trivial bounds but not for finite N_p . We checked that our calculations satisfy these bounds.

The direct knowledge of Ψ_{OD} allows for a last, very important consistency check. In fact, the value found from the Hellmann-Feynman theorem must equal, at half-filling, the sum $\frac{1}{N_p} \sum_{\alpha, \beta=1}^N \langle S_{\alpha}^+ S_{\beta}^- \rangle$ calculated from the determinant representation. We checked for all $N_p \leq 64$ that this is indeed the case.

VI. CONCLUSIONS

We have studied the static correlation functions of the reduced BCS model in the canonical ensemble. From the theoretical point of view, we simplified the results of Ref. 13, giving the correlations as sums over only N_r determinants of $N_r \times N_r$ matrices. This allowed us to calculate the correlation functions for very large numbers of particles, describing the crossover from mesoscopic to the thermodynamic limit and going beyond previous exact or approximate studies. In particular, with such accurate calculations, we were able to discuss critically some conjectured scaling forms for the canonical order parameter. For example, we rule out the idea of any phase transition as a function of the (positive) pairing strength and number of particles, in agreement with other analyses based on thermodynamical quantities.^{6,38,39} We also calculate the off-diagonal long-range order parameter by using the Hellmann-Feynman theorem.

A first interesting step to go beyond what has been done here would be to find a single determinant representation for the correlation functions. We made several attempts in this direction, but so far unsuccessfully.

We only analyzed the case of N nondegenerate equidistant energy levels at half-filling. This is the most interesting model from the condensed matter point of view. However, for the description of pairing in nuclei, other choices of the parameters ϵ_{α} are more natural.^{9,40,41,44} These could be treated by a simple adaptation of our results.

Furthermore, the method presented here allows, in principle, to obtain dynamical correlation functions as sums of form factors over the excited states. This is a more numerically demanding problem but can be tackled in the same way as other Bethe ansatz solvable models.^{42,43} Also, by summing

over the excited states, one can access the finite temperature thermodynamics and this can help in understanding some open questions^{44,45} for ultrasmall metallic grains.

The Hamiltonian in Eq. (1) is the simplest one with pairing terms. More general models with several couplings have been proposed to explain complicated pairing in condensed matter⁴⁶ and nuclear physics.⁴⁷ Some have been also shown to be integrable.^{14,48} As a consequence, it would be extremely interesting to tackle these models with methods similar to those presented here.

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